Nonlinear stability of vortex formation in swarms of interacting particles

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(Received 8 April 2008; published 29 July 2008)

We use a particle-based model of a swarm of interacting particles to explore analytically the conditions for the formation of vortexlike behavior. Our model uses pairwise interaction potentials to model weak long-range attraction and strong short-range repulsion with a dissipation function to align particle velocity vectors. We use the effective energy of the swarm as a Lyapunov function to prove convergence to a vortexlike state. Our analysis extends previous work which has relied purely on simulation to explore the formation and stability of vortexlike behavior through analytical rather than numerical methods.

DOI: 10.1103/PhysRevE.78.012903 PACS number(s): 87.23.Cc, 05.65.+b, 47.32.C−, 87.10.−e

Swarming patterns have been observed and reported for various species in nature [1]. The coherent flock and single-mill states are among the most common observed in biological swarms [2,3]. An example of a double-mill pattern, which is occasionally observed, has also been introduced [4]. Emerging vortex patterns among individuals that interact through pairwise artificial potential fields have been discussed by various authors [5–9]. In particular, we have been shown that the total linear and angular momenta of the swarm are conserved with a pairwise dissipation function [9]. The vortex pattern was then shown to be a constrained minimum of the total effective energy of the swarm. While it was shown that the vortex pattern was an extremum of the total effective energy, stability was not addressed. In the work reported here we use a Lyapunov function to demonstrate that the swarm will always relax into a vortexlike state. This analytic approach extends previous work on vortexlike behavior in particle-based models which have relied purely on simulation [8] and provides analytical insights into results from heuristic rule-based simulation [10]. In addition, through the use of analytic methods, our work has wider application to the construction of provable behaviors in swarms of interacting robotic agents.

We consider a swarm that consists of N identical particles of equal mass m with position and velocity $(\mathbf{x}_i, \mathbf{v}_i)$ defining the state of the ith particle. Attraction among the particles in the swarm is defined through a weak long-range attractive potential $U^a{}_{ij} = -C_a \exp(-|\mathbf{x}_{ij}|/l_a)$, while collisions between particles are prevented through a strong short-range repulsive potential $U^r{}_{ij} = C_r \exp(-|\mathbf{x}_{ij}|/l_r)$ [5–7]. The strengths of the attraction and repulsion potentials are denoted by C_a and C_r with ranges l_a and l_r, respectively. The particles attempt to align their motion with neighbors through a velocity-dependent orientation force $\mathbf{\Lambda}_i$, which is defined as $\mathbf{\Lambda}_i = \Sigma_{j\neq i} C_o(\mathbf{v}_{ij}\cdot\hat{\mathbf{x}}_{ij})\exp(-|\mathbf{x}_{ij}|/l_o)\hat{\mathbf{x}}_{ij}$, where $(\hat{\cdot})$ denotes a unit vector, C_o is the strength of the orientation force, and l_o is the range of the orientation force. Parallel orientation of the particle velocity vectors then emerges due to the dissipative nature of the orientation force such that motion towards or away from neighbors is weakly damped, proportional to the component of relative velocity along the vector connecting neighboring particles, $\mathbf{v}_{ij}\cdot\hat{\mathbf{x}}_{ij}$. This pairwise dissipation therefore results in a local alignment of particle velocity vectors, as used extensively in heuristic rule-based approaches [10]. The exponential term in the orientation force ensures that the effect is localized, while the pairwise interaction along $\hat{\mathbf{x}}_{ij}$ leads to conservation of angular momentum. We note, however, that due to the summation over all particles in the swarm, there is an inherent bias towards the behavior of the group rather than solely discrete pairs of particles.

The evolution of the swarm of interacting particles is now defined through the interaction potential and orientation force such that

$$\dot{\mathbf{x}}_i = \mathbf{v}_i, \tag{1a}$$

$$m\dot{\mathbf{v}}_i = -\nabla U_i^a - \nabla U_i^r - \mathbf{\Lambda}_i, \tag{1b}$$

where $U_i = \Sigma_j U_{ij}$ and $\nabla(\cdot) = \partial(\cdot)/\partial\mathbf{x}_i$. The three terms in Eq. (1b) are defined such that $l_r < l_o < l_a$. This arrangement is equivalent to the zone of repulsion, zone of orientation, and zone of attraction which has been used successfully in both rule-based simulation [10] and laboratory experimentation with biological swarms [11]. The use of artificial potential fields to mediate interactions between particles provides a continuous representation of these rule-based methods, which, unlike rule-based heuristics, is amenable to analytic investigation and formal proof.

The effective total energy of the swarm ϕ is now defined through a summation to evaluate each pairwise potential interaction and a summation of the kinetic energy of each particle. Therefore, the total effective energy of the swarm is defined as

$$\phi = \frac{1}{2}\sum_i m\mathbf{v}_i^2 + \sum_i (U_i^a + U_i^r). \tag{2}$$

Taking the time derivative of Eq. (2), it can be seen that

$$\dot{\phi} = \sum_i \mathbf{v}_i \cdot (m\dot{\mathbf{v}}_i + \nabla U_i^a + \nabla U_i^r). \tag{3}$$

Then, substituting from Eq. (1b) into Eq. (3), it can further be seen that

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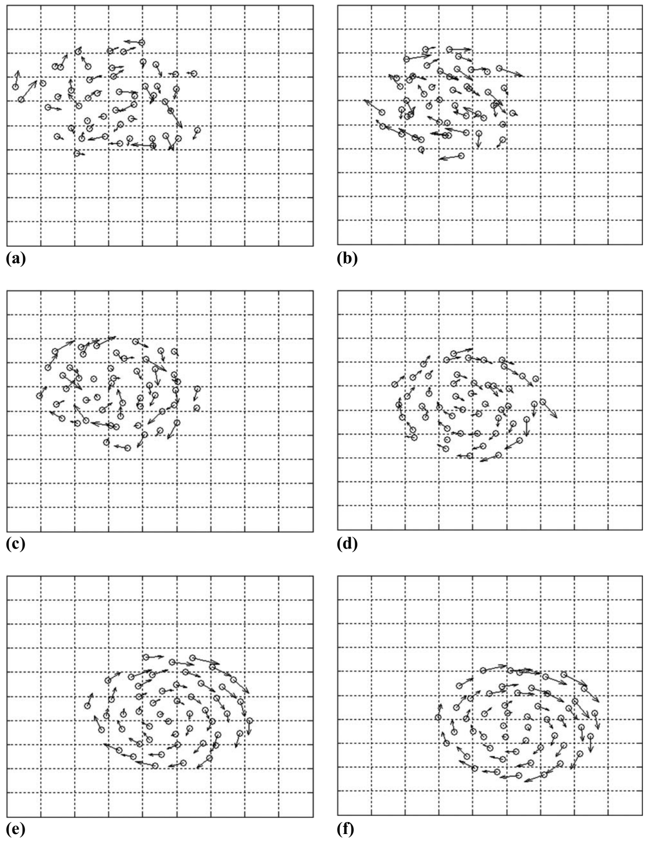

FIG. 1. Formation of a vortexlike pattern in a swarm of interacting particles (N=50) with C_a=1, C_r=2, C_o=0.1, l_a=1, l_r=0.2, and l_o=0.5 for nondimensional time t=0 until t=7 (top left to bottom right).

$$\dot{\phi} = -\sum_i \mathbf{v}_i \cdot \mathbf{\Lambda}_i \tag{4}$$

and so

$$\dot{\phi} = -\sum_i \mathbf{v}_i \cdot \sum_{j\neq i} C_o(\mathbf{v}_{ij}\cdot\hat{\mathbf{x}}_{ij})\exp(-|\mathbf{x}_{ij}|/l_o)\hat{\mathbf{x}}_{ij}. \tag{5}$$

We now demonstrate that $\dot{\phi}<0$ by considering an arbitrary term in the summation as

$$S_{ij} = (\mathbf{v}_i\cdot\hat{\mathbf{x}}_{ij})(\mathbf{v}_{ij}\cdot\hat{\mathbf{x}}_{ij})\exp(-|\mathbf{x}_{ij}|/l_o) + (\mathbf{v}_j\cdot\hat{\mathbf{x}}_{ji})(\mathbf{v}_{ji}\cdot\hat{\mathbf{x}}_{ji})\exp(-|\mathbf{x}_{ji}|/l_o). \tag{6}$$

However, noting that $\hat{\mathbf{x}}_{ji}=-\hat{\mathbf{x}}_{ij}$ and $\mathbf{v}_{ji}=-\mathbf{v}_{ij}$, it can be seen that

$$S_{ij} = (\mathbf{v}_i\cdot\hat{\mathbf{x}}_{ij})(\mathbf{v}_{ij}\cdot\hat{\mathbf{x}}_{ij})\exp(-|\mathbf{x}_{ij}|/l_o) - (\mathbf{v}_j\cdot\hat{\mathbf{x}}_{ij})(\mathbf{v}_{ij}\cdot\hat{\mathbf{x}}_{ij})\exp(-|\mathbf{x}_{ij}|/l_o), \tag{7}$$

and so using the identity $\mathbf{v}_{ij}=\mathbf{v}_i-\mathbf{v}_j$, it can further be seen that

$$S_{ij} = (\mathbf{v}_{ij}\cdot\hat{\mathbf{x}}_{ij})^2\exp(-|\mathbf{x}_{ij}|/l_o). \tag{8}$$

The rate of change of the total effective energy of the swarm can therefore be written as

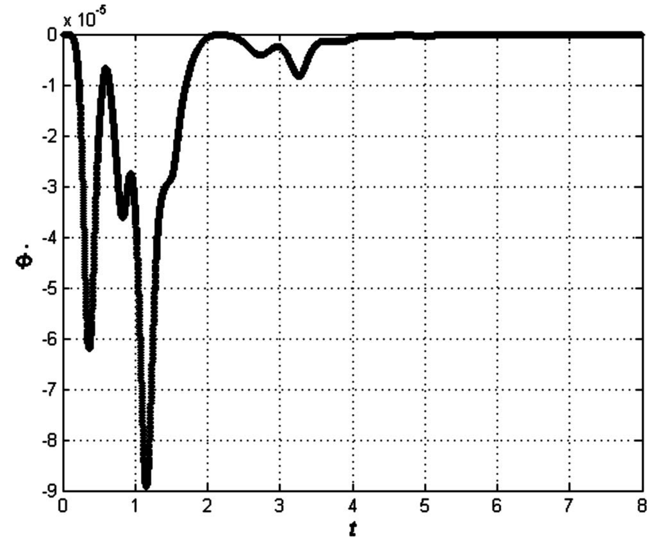

FIG. 2. Time rate of change of the total effective energy of the swarm.

$$\dot{\phi} = -\frac{1}{2}\sum_{i=1}\sum_{j\neq i} C_o(\mathbf{v}_{ij}\cdot\hat{\mathbf{x}}_{ij})^2\exp(-|\mathbf{x}_{ij}|/l_o). \tag{9}$$

Since $C_o \geqslant 0$, the quadratic term in Eq. (9) ensures that $\dot{\phi}<0$ so that the total effective energy of the swarm is monotonically decreasing.

In previous work we demonstrated that vortexlike patterns could be interpreted as a constrained minimum-energy state [9]. Considering the total effective energy of the swarm,

$$E = \left(\frac{1}{2}\sum_i m\mathbf{v}_i^2 + \sum_i (U_i^a + U_i^r)\right) - \boldsymbol{\lambda}\cdot\left(\sum_i m\mathbf{x}_i\times\mathbf{v}_i - \mathbf{H}\right),$$

and enforcing conservation of total angular momentum $\mathbf{H}$ through a Lagrange multiplier λ, it was shown that

$$\frac{\partial E}{\partial \mathbf{x}_i} = (\nabla U_i^a + \nabla U_i^r) - m\boldsymbol{\lambda}\times\mathbf{v}_i = \mathbf{0}, \tag{10a}$$

$$\frac{\partial E}{\partial \mathbf{v}_i} = m(\mathbf{v}_i - \boldsymbol{\lambda}\times\mathbf{x}_i) = \mathbf{0}, \tag{10b}$$

so that the constrained minimum-energy state of the swarm corresponds to vortexlike rotation with the velocity vector of each particle normal to its position vector and the vector λ such that $\mathbf{v}_i=\boldsymbol{\lambda}\times\mathbf{x}_i$. The Lagrange multiplier λ was identified as the angular velocity vector of the swarm which is directed along $\mathbf{H}$. Therefore, it can be seen that in the constrained minimum-energy state $\mathbf{v}_{ij}=\boldsymbol{\lambda}\times\mathbf{x}_{ij}$ and so, in Eq. (9), $\mathbf{v}_{ij}\cdot\hat{\mathbf{x}}_{ij}=(\boldsymbol{\lambda}\times\mathbf{x}_{ij})\cdot\hat{\mathbf{x}}_{ij}$. However, using the scalar triple product identity $\mathbf{v}_{ij}\cdot\hat{\mathbf{x}}_{ij}=\boldsymbol{\lambda}\cdot(\mathbf{x}_{ij}\times\hat{\mathbf{x}}_{ij})=0$, and so $\dot{\phi}=0$ in the vortexlike state.

We have therefore demonstrated that the vortexlike state is an extremum of effective swarm energy and that this extremum is a global minimum by use of a Lyapunov function. It can therefore be concluded that with the orientation force $\mathbf{\Lambda}_i$, a swarm of particles in an initially random state will always relax into a spatially coherent vortexlike pattern, as observed in a wide range of biological swarms [1,3,12] and in simulation [5,7,10,13]. The emergence of the stable vortexlike state depends only on the dissipation force and not on the specific structure of the weak long-range attraction and strong short-range repulsion potentials. Again, we note that the use of artificial potential fields to mediate interactions between particles provides a continuous representation of rule-based methods with the length scales $l_r < l_o < l_a$ equivalent to the zone of repulsion, zone of orientation, and zone of attraction used in rule-based simulation [10] and laboratory experimentation [11].

Finally, in order to illustrate the formation of vortexlike patterns using the mechanism discussed above, a planar swarm of $N=50$ particles is considered. The particles in the swarm are randomly distributed over a unit disk with a random distribution of initial velocities. The free parameters are selected such that $l_r < l_o < l_a$ so that the swarm experiences weak long-range attraction, strong short-range repulsion, and local velocity alignment. It can be seen from Fig. 1 that the swarm slowly relaxes into a vortexlike pattern. As the swarm relaxes, the time rate of change of the total effective energy of the swarm vanishes, as shown in Fig. 2.

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